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Number Determiners, Numbers, and Arithmetic

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1. Frege's Other Puzzle

In his groundbreaking *Grundlagen*, Frege (1884) pointed out that number words like 'four' occur in ordinary language in two quite different ways and that this gives rise to a philosophical puzzle. On the one hand 'four' occurs as an adjective, which is to say that it occurs grammatically in sentences in a position that is commonly occupied by adjectives. Frege's example was

(1) Jupiter has four moons,

where the occurrence of 'four' seems to be just like that of 'green' in

(2) Jupiter has green moons.

On the other hand, 'four' occurs as a singular term, which is to say that it occurs in a position that is commonly occupied by paradigmatic cases of singular terms, like proper names:

(3) The number of moons of Jupiter is four.

Here 'four' seems to be just like 'Wagner' in

(4) The composer of Tannhäuser is Wagner,

and both of these statements seem to be identity statements, ones with which we claim that what two singular terms stand for is identical.

But that number words can occur both as singular terms and as adjectives is puzzling. Usually adjectives cannot occur in a position occupied by a singular term, and the other way round, without resulting in ungrammaticality and nonsense. To give just one example, it would be ungrammatical to replace 'four' with 'the number of moons of Jupiter' in (1):

(5) *Jupiter has the number of moons of Jupiter moons.

This ungrammaticality results even though 'four' and 'the number of moons of Jupiter' are both apparently singular terms standing for the same object in (3). So, how can it be that number words can occur both as singular terms and as adjectives, while other adjectives cannot occur as singular terms, and other singular terms cannot occur as adjectives?

Even though Frege raised this question more than one hundred years ago, I dare say that no satisfactory answer has ever been given to it. Some attempts to answer it are lacking in a number of ways, and in this article I hope to make some progress toward an answer to this puzzle. Since Frege first raised the puzzle, it might be called Frege's Puzzle, but that term is already reserved for the puzzle that Frege also raised about identity statements and belief ascriptions, which is unrelated to our puzzle here (see, for example, Salmon 1986). I will thus call the puzzle about the different uses of number words *Frege's Other Puzzle*. Frege's Other Puzzle is, strictly speaking, only a puzzle about natural language, but its importance goes beyond that. I described it as a puzzle about grammar and syntax, but it quickly turns into a puzzle about the semantic function of number words as well. Singular terms paradigmatically have the semantic function of standing for an object, whereas adjectives paradigmatically modify a noun and do not by themselves stand for objects.[1] If number words fall into one or the other of these categories, then this will be of great interest for the philosophy of mathematics. If number words are singular terms that stand for objects, then arithmetic presumably is a discipline about these objects. But if number words are adjectives that do not stand for objects, then arithmetic will have to be understood along different lines. Whether arithmetic is a discipline that aims to describe a domain of objects or does something else is a question that can be closely associated with the question what the semantic function of number words is. Frege's Other Puzzle is thus not only a puzzle about the syntax and semantics of natural language, but is also of great interest for the philosophy of mathematics.

To be more precise, we can distinguish the simple version of Frege's Other Puzzle, which is the puzzle about the different uses of number words in natural language, from the extended version of Frege's Other Puzzle. The extended version covers not only number words in natural language, but also symbolic numerals, like '4', which are pronounced just the same way as number words are pronounced in natural language. These symbolic numerals are the ones used in mathematics proper.[2] How they relate to number words in ordinary everyday language is a question that leaves room for some debate. One simple, though not implausible, view is that symbolic numerals are merely abbreviations of natural language number words. But if so, which uses of number words do they abbreviate? Let's call a *uniform solution* to the simple version of Frege's Other Puzzle a solution according to which, in natural language, the very same number word can occur either as a singular term or as an adjective. Such a solution will have to explain how one and the same word can occur in these two different ways. And let's call a *nonuniform solution* one where they are not one and the same word. Such a solution will have to explain how these different words relate to each other. If we have a uniform solution to the simple version of Frege's Other Puzzle, then it will plausibly extend to a solution to the extended version of Frege's Other Puzzle. If, in natural language, one and the same number word can occur in these two different ways, then it is reasonable to think that symbolic numerals are abbreviations of natural language number words. This is plausible, but not guaranteed to be so. After all, it could be that mathematical uses are different from ordinary natural language uses. But that the symbols and the number words are pronounced the same way doesn't seem to be a mere coincidence. Thus, it is not unlikely that symbolic uses of numerals are derivative either on the singular-term use or the adjectival use of number words, and that they (symbolic numerals) abbreviate one or the other in symbolic notation. But the interaction between these words and symbols could also be more complex. It could be that the symbolic uses of numerals have an effect on the uses of number words in natural language. In fact, we will closely explore this possibility below.

We can thus distinguish at least three different uses of number words. The *singular-term* use, as in (3), the *adjectival*, or as I will also call it from now on, for reasons to be explained shortly, the *determiner* use, as in (1), and the *symbolic* use, as in '4'. The main question for the following is how they relate to each other.

2. How the Puzzle Can't Be Solved

One might think that one can avoid all this difficulty quite easily by simply claiming that 'four' is ambiguous and thus there is one word that is an adjective and another that is a singular term, both happen to be spelled the same way. This would avoid the difficulty how one and the same word can occupy two different syntactic positions and have two different semantic functions. But this by itself is no solution to our problem. The singular-term use and the determiner use of 'four' are not independent and unrelated. (1) and (3) are clearly closely related; in fact they seem to be quite obviously equivalent. But how could this be if 'four' is ambiguous, like 'bank', with different meanings in (1) and (3)? Simply to say that they are ambiguous is not enough to explain how the two uses relate to each other. In addition, every number word would be ambiguous in just the same way: 'one', 'two', 'three', and so forth. Such a systematic ambiguity will have to be explained, and by itself can't be the answer. What we need is an account of how and why number words are systematically used in at least two different ways, whether or not they are ambiguous. The claim that these words are really two different words pronounced the same way does not help us understand this.

There is a long-standing tendency in the philosophy of mathematics to discard the adjectival or determiner uses of number words. There are two main lines of argument to justify this; one is very widespread, the other one less so. The less important line sees adjectival uses of number words as merely a curious feature of natural language, not something to be taken too seriously, in particular by those who are mainly concerned with science and mathematics. This goes back at least to Frege. In fact, this is what he suggests shortly after pointing out in the *Grundlagen* that number words can occur with apparently two different syntactic and semantic functions. Frege says:

> Now our concern here is to arrive at a concept of number usable for the purposes of science; we should not, therefore, be deterred by the fact that in the language of everyday life number appears also in attributive constructions. That can always be gotten round. For example, the proposition 'Jupiter has four moons' can be converted into 'The number of moons of Jupiter is four.'[3] (Frege 1950, §57)

Of course, it is not completely clear what Frege's considered judgment is on these issues, even if we consider only the *Grundlagen*, and this article is not the place to settle questions about Frege. But Frege clearly seems to give primacy to the singular-term uses of number words, and he holds numbers to be objects. The above passage seems to suggest that the adjectival or attributive uses of number words can be put aside for a more serious investigation since they are merely an avoidable feature of everyday language and not to be expected in the language that will be suitable for science. And this seems to suggest that adjectival uses of number words are a feature of natural language that would not be found in a more ideal language suitable for science. The attributive uses of number words are thus cases where an ideal language and natural language come apart.

Whatever Frege's considered judgment is on these issues, we should not be satisfied with the answer outlined above. First of all, it is not clear why scientific language should not also partake in determiner uses of number words. Why is such use not to be taken at face value? Even if it can be avoided, why should we avoid it? And why should we avoid the adjectival use, and not the singular-term use? The singular-term use often can't be paraphrased away with just the adjectival use, but we will see cases below where the opposite is true as well, that is, adjectival uses that can't be paraphrased as singular-term uses. In addition, this attitude does not help use to solve Frege's Other Puzzle since the latter is after all about natural language. Putting aside natural language won't help us here, even if we could paraphrase all determiner uses away.

A further, more important and widely attempted way to deal with Frege's Other Puzzle, or at least to get around it, is what I will call the *syncategorematic account*. According to this proposal, determiner uses of number words are to be understood as syncategorematic; they disappear upon analysis. This proposal is inspired by a proposal that Russell made about the word 'the' in his theory of descriptions, and it can be motivated as follows: A number word can be combined with a noun to form a numerical quantifier. Such quantifiers, the syncategorematic analysis goes, can be understood as complexes of the quantifiers '$\exists$' and '$\forall$'. These quantifiers are part of the first-order predicate calculus, and this calculus, in turn, is part of logic, and thus unproblematic. Take for example

(6) A man entered,

which contains a quantifier, and which could semiformally be written as

(7) $\exists x : x$ is a man and x entered.

Similarly,

(8) Two men entered

can be understood as involving more than one quantifier. Semiformally it could be written as

(9) $\exists x \exists y : x \neq y$ and x is a man and y is a man and x entered and y entered.[4]

And so on for other number words. Thus, at the level of semantic representation, the number words disappear when they are used as determiners. What is left are blocks of first-order quantifiers.[5,6] And in this way the proposal is like Russell's proposal about descriptions. According to Russell, as he is usually understood, the word 'the' in a description is syncategorematic and disappears upon analysis (see Russell 1905). It doesn't make an isolatable, single contribution to the truth conditions. Rather, it is analyzed away in context. According to Russell, the underlying form of

(10) The man entered[7]

can be revealingly spelled out with the semiformal

(11) $\exists x : x$ is a man and x entered and $\forall y$ if y is a man then $y{=}x$.

The syncategorematic analysis thus attempts to solve Frege's Other Puzzle by proposing that determiner uses of number words disappear in the semantic analysis, whereas singular-term uses do not. Semantically, number words can thus be understood as having the function of standing for objects. That is to say, all number words that are still left at the level of "logical form" have the function of standing for objects. Number words in their determiner use only appear in that position in the surface syntax. They will have disappeared into blocks of first-order quantifiers at the level of logical form.

This proposal for solving Frege's Other Puzzle has a variety of problems. First, it works at most semantically, but not syntactically. There is no explanation why the word 'four' syntactically occurs as both a determiner as well as a singular term. After all, the word 'the' never occurs as a singular term. According to Russell's proposal, the word 'the' is of a fixed syntactic category, but it disappears at the level of semantics into blocks of first-order quantifiers. The word 'four', according to the syncategorematic account, disappears at the semantic level in its determiner uses, but not in its singular-term uses. The syntactic issue of how apparently one and the same word can occur in these different syntactic positions is not answered by this account.

Secondly, the puzzle that we are concerned with isn't answered by this proposal. We want to know how apparently one and the same word can do these different things: be used as a singular term and also be a part of quantifiers, or more generally how these different uses of 'four' relate to each other. Whether or not number words as determiners disappear at the level of logical form, the puzzle remains how they can both be singular terms and determiners. How can they sometimes stand for objects, but also sometimes syntactically occur as determiners or adjectives, and then semantically disappear with only blocks of quantifiers as a trace?

Finally, the view that words like 'the' or number words as determiners should be understood as syncategorematic should be rejected for completely different reasons. These reasons are widely discussed, and accepted, in the case of the syncategorematic treatment of 'the' that Russell proposed. But they are not as widely appreciated in the philosophy of mathematics literature, where the syncategorematic treatment of number determiners is often endorsed, even though these reasons apply there equally well.

Words like 'the', 'four', 'some', 'many', 'most', and so forth, are rather similar. They combine with nouns to make full noun phrases, and they all can form similar sentences, like

(12) The F is G.

(13) Some F is G.

(14) Most F are G.

However, according to the syncategorematic analysis, some of them are syncategorematic, others are not syncategorematic, and others again are not treated by this analysis at all. 'The' is supposed to be syncategorematic, whereas 'some' is not; it gets represented directly as a first-order quantifier at the level of logical form. 'Most', on the other hand, can't be treated syncategorematically along the lines discussed above since it is known not to be first-order expressible. By "first-order expressible" I mean "expressible in the first-order predicate calculus with the quantifiers '$\exists$' and '$\forall$'." The syncategorematic analysis only gets off the ground since some determiners are first-order expressible. The quantifiers that can be formed with them can be expressed with complexes of first-order quantifiers as well. Other determiners, like 'most' can't be expressed this way.

But what is the relationship between a natural language sentence and its predicate calculus counterpart, even in cases where the truth conditions can be expressed this way? Does the first-order sentence reveal the underlying logical form, or is it merely another way to express the same truth conditions? As we will see in just a minute, a unified semantics of all determiners can be given. It takes to heart that the first-order predicate calculus is not expressive enough to properly treat natural language quantification, and it will assign the same underlying structure to all of our above examples (ignoring issues about plural and singular for now). The syncategorematic treatment of number determiners would assign completely different underlying logical forms even to

(15) Four F are G

and

(16) Two hundred F are G.

The latter would have two hundred first-order quantifiers in its logical form. Since a semantics for determiners can be given that gets the truth conditions right and doesn't have any of the flaws of the syncategorematic analysis, we will have to conclude that the syncategorematic analysis gives merely another way to express the same truth conditions in first-order logic, but not one that is semantically revealing and that brings out the underlying logical form.[8]

The syncategorematic analysis thus has to be rejected. A closer look at the semantics of number determiners in the semantic theory that replaced the syncategorematic one is, however, most useful for getting closer to solving Frege's Other Puzzle. Thus we will look at it in some more detail next.

3. Number Determiners

3.1 Determiners and Quantifiers

A view of natural language quantification that is closely tied to first-order quantification, that is quantification in the predicate calculus with the quantifiers '$\forall$' and '$\exists$', has several limitations. First, first-order quantifiers take a full noun phrase to be the basic unit of analysis. Thus 'something' is the smallest unit of a quantifier, analyzed as '$\exists x$'. However, natural language quantifiers are often complex and built up from smaller units. In fact, even 'something' seems to be built up from 'some' and 'thing'. Natural language quantifiers contain phrases such as 'most men', 'every tall child', and so forth. It would be nice to have a view that specifies the semantics of a complex quantifier on the basis of the semantics of its simpler parts. That is, it would be nice to have a compositional semantics of complex quantified noun phrases, which includes a semantic treatment of the smaller parts. Secondly, many natural language quantifiers can't, for logical reasons, be understood as first-order quantifiers, or complexes thereof. These quantifiers are provably not first-order expressible, like 'most men'. But a unified account of natural language quantification should treat all quantifiers, not just the ones expressible in a first-order language.

These requirements have been met by generalized quantifier theory (GQT), a very successful and widely accepted theory of natural language quantification. It is based on the work of Mostowski (see Mostowski 1957) and Montague (see Montague 1974) and has been developed by various logicians and semanticists.[9] We can't get into the details of this theory, but the basics are simple enough to state and are sufficient for our purposes here.

Natural language quantifiers have an internal structure whose basic parts are a noun and a determiner. Focusing on the simplest case, nouns are words like 'man', 'thing', and so forth, and determiners are words like 'some', 'all', 'many', 'most', 'the', and many more, including number words in their determiner use. The semantics of these quantifiers is compositionally determined by the semantics of the noun and the determiner. The appropriate semantic values for these expressions are "higher-order objects," usually understood as follows. The semantic value of a sentence is a truth value. A sentence splits up into a noun phrase (NP) and a verb phrase (VP). The semantic value of a VP is a property, and the semantic value of a NP is a function from properties to truth values. The semantic value of a sentence is thus determined by the semantic values of its immediate parts: simply apply the function that is the semantic value of the NP to the semantic value of the VP. We will ignore the internal structure of VPs here, but we will look at the internal structure of a quantified NP. It splits up into a determiner and a noun. The semantic value of a noun can be a property as well, and the semantic value of a determiner is thus a function from properties to functions from properties to truth values. Thus the semantic value of a determiner is a function from semantic values of nouns to semantic values of full NPs. In a notation used to specify these higher type objects, an object of the type of truth value is of type t, an object of the type of an entity is type e, and a function from one type to another is written as (t_1, t_2). If we understand properties to be functions from entities to truth values,[10] determiners are thus of type $((e, t), ((e, t), t))$. These types can be understood semantically as well as syntactically. Semantically, they specify what kind of a semantic value an expression has. Syntactically, they specify how an expression combines with others to ultimately form a sentence, which is of type t.

As it turns out, this theory works perfectly well, at least for the cases we are considering here, and it is widely accepted. It gives us a unified compositional semantics for quantified noun phrases and, in fact, other noun phrases as well. Different determiners just get assigned different functions, but all get a semantic value of the same kind.

For our discussion in this article, there are two important points to note here. First, according to GQT the word 'two' (in its determiner use) is not syncategorematic. It makes a discernible contribution to the truth conditions and is not explained away at the level of logical form. Secondly, number determiners, that is, number words in their determiner use, have a semantic value, just like all determiners, but they are not referring expressions or singular terms. Having a semantic value and having a referent have to be distinguished. Every expression has a semantic value and might have a different one in different semantic theories. But not every expression is a referring expression. Names and some other singular terms refer, but 'some' and 'many' don't. In addition, even referring expressions can have a semantic value other than their referent. In Montague's treatment of proper names, for example, names get sets of properties (or, equivalently, functions from properties to truth values) as their semantic value, but they don't refer to such functions. They refer to people, dogs, and the like. Thus, even though determiners like 'some', 'many', and 'two', have certain semantic values in GQT, they do not refer to these semantic values.[11, 12]

That number determiners are meaningful expressions that are not singular terms and that are not syncategorematic must be beyond question, given the success and the wide acceptance that generalized quantifier theory enjoys and the obvious advantages it has over the alternatives we have discussed. It was not beyond question when Frege wrote the *Grundlagen*, and maybe not when early neo-Fregeans remarked that such uses of number words are ultimately singular terms (see Wright 1983 for such remarks). But today this view cannot be accepted, and it at least has to be taken as a radical proposal about natural language quantifiers. I stress this to make vivid that we have then no solution available to Frege's Other Puzzle, either in its simple or extended version. Generalized quantifier theory does not cover number words when they occur as singular terms, only when they are determiners. We thus have theories that deal with one or the other of the two occurrences, but none that gives a unified account of both.

In the following, I will propose a further account, one that starts with ordinary uses of number determiners and that, together with some empirical considerations, attempts to provide a unified account of all uses of number words. Not all aspects of this proposal are fully contained in this article, and some are subject to empirical confirmation or refutation. But the proposal to follow gives us a new solution to Frege's Other Puzzle. Given the failure of the proposals outlined above, we badly need to try something else. The key to a different account of the various uses of number words is to take the determiner uses more seriously and not to brush them aside as syncategorematic. In fact, the determiner uses are quite a bit more interesting and complex than one might think.

3.2 Bare Determiners

As a determiner, 'two' takes a noun argument to make a full noun phrase. Determiners can also occur in sentences without their arguments being made explicit. Consider

(22) Three men entered the bar, and two stayed until sunrise.

(23) Every man entered the bar, but only some stayed until sunrise.

(24) Three eggs for breakfast is too many, two is about right.

In all these examples, the last occurrence of a determiner in these sentences is without an argument, at least without it showing up explicitly in the sentence. This is not too puzzling and easily explained as ellipsis. The argument was just mentioned half a sentence ago and isn't repeated again. There are, however, a number of examples where the determiner also occurs without the argument being explicit, but where it does not seem to be a case of ellipsis. Consider:

(25) Some are more than none.

(26) Many are called upon, but few are chosen.

(27) None is not very many.

(28) All is better than most.

Such examples (except maybe (26)) can be used either elliptically or to make a general statement. When (25), for example, is used to make a general statement, it basically means that, of whatever, some are more than none. The elliptical use of (25) means that a contextually salient kind of thing will be such that one claims that some of them are more than none of them. I am using the term 'elliptical' broadly here to include the case of a contextually salient but not explicitly mentioned kind. This phenomenon also occurs with number determiners. They can be used elliptically, as above, or to make a general statement, as in standard utterances of

(29) Five are more than three.

(30) Two are at least some.

Let's call (an occurrence of) a determiner without explicit argument or noun a *bare determiner.* As we saw, we have to distinguish between two kinds of bare determiners, the ones that are like the elliptical ones, and the ones that are used to make general statements. Let's call the elliptical ones *elliptically bare determiners*, keeping in mind our broad use of the term 'elliptical', and the other ones *semantically bare determiners.* The distinction is supposed to be exclusive and applies to occurrences of determiners in particular utterances. Another example of the first kind are utterances of

(31) After dinner I will have one, too.

The kind of thing of which the speaker said he will have one will be fixed by the context of the utterance. So, the context will fix a certain kind of thing, X, such that what the speaker said is true just in case the speaker has one X after dinner. So, 'one' is an elliptically bare determiner here. Another example of semantically bare determiners is an ordinary use of

(32) Two are more than none,

where the speaker is not intending to talk about any particular kind of thing.

3.3 Complex Determiners and Quantifiers

Before we can address issues more directly related to Frege's Other Puzzle, we should also look at operations on determiners and quantifiers. Natural language allows us to build more complex determiners out of more simple ones. The best examples of this are Boolean combinations of determiners as in

(33) Two or three men entered the bar.

(34) Some but not all women smoked.

And we can build more complex quantifier phrases out of more simple ones. Boolean combinations are again a good example:

(35) Some men and every woman smoked.

(36) Two men and three children saw the movie.

Such operations can also occur on bare determiners:

(37) Two or three is a lot better than none.

(38) Few or many, I don't care, as long as there are some.

Consider now

(39) Two apples and two bananas make a real meal.

(40) Two apples and two bananas are only a few pieces of fruit.

(41) Two apples and two bananas are four pieces of fruit.

Here 'Two apples and two bananas' form a complex quantified NP that consists of two quantified NPs that are joined together by 'and'. The semantic function of 'and' in these examples is worthy of a few words here. It can't be seen as merely the familiar sentential conjunction. In these uses 'and' combines quantifiers to make more complex quantifiers, and it thus conjoins quantifiers, not sentences. In particular, we are dealing with plural NPs here, and operations on plural NPs usually allow both so-called distributive and collective readings, just like plural NPs in general. To illustrate this, consider

(42) Three men carried a piano.

This could be read either collectively, where three men together, as a group, carry a piano, or distributively, where three men each carry a piano. Similarly,

(43) Three men and two women carried a piano,

could be read in a variety of ways: a group of three men carried one piano, and a group of two women carried another; or all five carried one piano; or each one of them carried one piano; and maybe others as well. This distinction is important when it comes to understanding cases like

(44) Pizza and cheap beer make me sick,

which might or might not imply

(45) Pizza makes me sick and cheap beer makes me sick,

depending on whether or not 'and' is read collectively. In particular, if it is read collectively, then the 'and' involved can't be understood as a truth-functional sentential connective.

Just as there are sentences that involve basically only bare determiners, like our examples (25), (27), and (28) above, there are also sentences that involve basically only bare determiners, but these determiners are complex. For example, consider

(46) Two or three are at least some.

All this is pretty straightforward. But this gets us closer to something that almost sounds like a bit of arithmetic.

3.4 Arithmetic-like Bare Determiner Statements

Some statements that crucially involve bare determiners are apparently very close to arithmetic statements, though, as we will see below, there are also important differences among standard expressions of arithmetical statements. Consider:

(47) Two and two are four.

Here we have three bare determiners at once in the sentence, the first two combined by 'and' (in its collective reading). In most uses, these determiners will be semantically bare. Note also that (47) is plural, as the bare determiner use would seem to require. These statements not only sound a lot like arithmetic, in fact they are quite close to it, as we will see in the following. However, before we can consider arithmetic, we should look at these bare determiner statements in some more detail.

(47) is a general statement involving semantically bare determiners. What it says can be spelled out as: two X and two (more) X are four X, for whatever X. In fact, there is another option that is equally good for our subsequent discussion, and which, in the following, I won't carefully distinguish from the reading just mentioned. It involves understanding 'and' as an operation on determiners, not NPs. So, according to this reading, (47) can be understood as two and two (more) X are four X, whatever X is. In the following discussion, either of these two readings will do. Note also that the qualification 'more' does not usually have to be made explicit; 'and' is mostly read as 'and in addition', not just in the above examples, but in many similar cases. Consider:

(48) She only had an apple and dessert.

A usual utterance of this wouldn't be true if she just had an apple, even though a fruit is a perfectly fine dessert. There are a variety of mechanisms that can guarantee this to be so. It could simply be ellipsis, or a pragmatic mechanism, or a form of "free enrichment," or something else, but which is the correct one in cases like this does not matter for our discussion here.

How should we think about statements like (47) with more large-scale philosophical issues in mind? First of all, (47) contains no referring or denoting expressions. The words 'two' and 'four' occur in their determiner use, and as determiners they are not referring expressions. They are part of NPs, but not by themselves NPs. In particular, these words do not refer to determiners, just as 'some' or 'none' do not refer to determiners in (25). In both of these statements, no reference occurs. The truth of these statements, thus, does not depend on the existence of any objects that are referred to since none are referred to. Indeed, these statements are true no matter which or how many objects exist, as we will see below. Whatever there might be, some are more than none, and two and two are four. I won't be able to argue for this here, but I would hold that, not only are these statements true, they are both objectively and necessarily true. It couldn't be the case that some are not more than none, or that two and two are not four. This, of course, deserves further support, but I will not be able to provide it here. I trust that, from what we have seen so far, it seems reasonable or at least tenable to hold this for the bare determiner statements. I will discuss an objection to this below.

4. Numbers and Arithmetic

We have seen that statements formulated with bare determiners can be a lot like arithmetical statements, and that they apparently can be true no matter what objects exist. But it would be quite premature to think that this can be easily extended into an account of arithmetic and talk about numbers in general. It is now time to take a closer look at how what we have seen so far relates to arithmetic and the theory of numbers. This will bring us closer to a new solution of Frege's Other Puzzle. It might not be completely clear why looking at arithmetic and number symbols is the natural thing to do if we want to solve Frege's Other Puzzle, since this puzzle, in its simple version, was about number words in natural language. But as I mentioned above, it might well be that arithmetic and symbolic numerals affect the uses of number words in natural language. A look at how the bare determiner statements relate to arithmetic will clarify this and lead to a solution of Frege's Other Puzzle, a solution that can't be seen by looking at natural language in isolation. Thus I will now discuss basic, simple arithmetical statements.

4.1 Basic Arithmetical Statements

Basic, quantifier-free arithmetic is a trivial part of mathematics, but it is philosophically important for at least two reasons. First, understanding it is an important step toward understanding arithmetic. Second, it is in many ways the first and most basic part of mathematics. It is the first mathematics a child learns, and it is most closely connected to ordinary nonmathematical discourse. Besides that, it is there where mathematical symbols are first introduced. A good starting point for understanding mathematics is thus trying to understand basic arithmetical statements. Among these is

(49) 2+2=4.

I use mathematical notation here since it is itself an important issue what these symbols really mean. What does (49) mean? How and why does this notation get introduced? A first step in answering the first question is simply to read (49) out loud. But, as it turns out, different people say different things (at different times) when asked to do so. When one asks different mathematically competent people to read (49) out loud, one gets at least the following answers (I report here on an informal survey):

(47) Two and two are four,

(50) Two and two is four,

(51) Two and two equal four,

(52) Two and two equals four,

and all the above with 'plus' instead of 'and'.[13] It seems that there are basically two ways to read '2+2=4', in the plural and in the singular. But on reflection, these two ways of looking at it seem to be quite different. As we have seen, when we speak in the plural and say

(47) Two and two are four,

we are using bare determiners and we are not talking about particular objects. But when we speak in the singular and say

(50) Two and two is four,

then it seems that we are saying something about particular objects. First of all, it seems that in (50) 'is' is the 'is' of identity. With it we are claiming that what one singular term, 'two and two', stands for is identical with what another singular term, 'four', stands for. And surely, for this to be true, the things these terms stand for have to exist. In addition, (50) interacts with quantifiers in exactly the way that seems to be required to establish that the terms 'two and two' and 'four' are referring expressions. In particular, we can infer that there is something which is four, namely two and two. So, it seems that (50) is an identity statement involving two singular terms that stand for objects, whereas (47) is a statement involving no referring terms, but only bare determiners.

The plural and singular ways of reading symbolic arithmetical equations are a striking case of Frege's Other Puzzle. The two sentences are as close to each other as possible, except that, in one, number words are used as singular terms and, in the other, they are used as determiners. These pairs of examples provide an interesting addition to Frege's pair (1) and (3). In both, the puzzle is about the relation of the different uses of the number words. In Frege's original pair, he claimed, and many but not all agreed, that they were equivalent in truth conditions. This seems less clearly so in our pair. As we have seen, on reflection the truth conditions seem to be quite different, and thus only one of them could be the correct way to read the mathematical equation, assuming it is unambiguous.

But is there really only one correct way to read the symbolic equation? To understand how the plural and the singular way of reading the symbolic arithmetical equations relate to each other would be to solve a special case of Frege's Other Puzzle. To understand this, we should not only try to find out which one of these is the correct reading of the symbolic equation, if there is only one correct reading, but also why there doesn't seem to be such a clear-cut answer to this. Why do many mathematically competent people say one or the other? We will have to look at how the arithmetical symbols are first introduced to us and what meaning is given to them to make progress on this. This can't be decided by looking at natural language or mathematics by themselves. We will have to look at how this actually happens when we learn arithmetic.

4.2 Learning Basic Arithmetic

How a child learns arithmetic is of considerable interest from many points of view. On the one hand, it is of interest to developmental psychologists, who take children's mastering of counting, arithmetic operations, and the number concepts as an important case study for large-scale issues in developmental psychology. On the other hand, it is of interest to educators who would like to understand better how children learn in order to improve the teaching of children. For these reasons, there is a substantial literature on how children learn mathematics, in particular arithmetic, and how this can be improved by using certain teaching methods. It seems to me that this literature (or at least the small part I know of) is full of hints about how to solve Frege's Other Puzzle.

In learning basic arithmetic children have to learn a number of different but connected things.[14] To name a few important ones:[15]

1. They have to learn or memorize how to continue the sequence: *one, two, three, four…*
2. They have to learn how to count down some collection of objects by associating with each object one of the first n numerals.
3. They have to be able to determine size, or answer "How many? " questions.
4. They have to master change of size, adding things to a collection or taking them away.
5. They have to master the mathematical formalism, like '2', '+' and so forth.
6. They have to learn how to solve arithmetical problems purely within the formalism, for example, give the right answer to the question: 2789+9867–34=?

How precisely this learning process goes temporally is not completely clear, though it seems to proceed in more or less the order in which they are listed above. Of course one does not have to reach perfection at one stage to go to the next. Kids don't first learn how to count all the way to 10^{10} and then learn how to add. First, kids have to learn how to count to, say, ten or twenty. Then they have to learn to give the right answers to questions like

(54) Do you see these cats in the picture, Johnny? How many are there?

This will be done by simply counting down the ones that one sees, at least if there are more than just very few. After that they will master judgments about changes of size, both in actual collections as well as in imagined ones. There are a variety of exercises used to do this, and a standard text book on teaching mathematics to first graders should contain a collection of a good mixture of them.[16] Examples are

(55) Here we have three marbles on the floor. Now I put two other ones there. How many marbles are now on the floor?

(56) Suppose Johnny has two marbles and Susie has three more than Johnny, how many does Susie have?

During this learning process, which takes quite some time, the teacher will introduce the mathematical symbols. The students will learn the decimal system, that ‘2’ is read ‘two’, and to count in symbols—that is, they will learn to continue the sequence *1, 2, 3, 4,* … And the student will learn to represent what was learned in exercises like (55) and (56) in symbols. After doing exercise (55), a classroom situation might continue:

(57) That’s right, Susie, three marbles and two more make one, two, three, four, five marbles. (*The teacher will write on the blackboard:* 3+2=5 *and say out loud any of the following:* Three and/plus two is/are/makes/make five.)

After children master such exercises to a reasonable degree, they will be taught to add, subtract, and multiply using the symbols alone. At this stage, children will learn tricks for adding that are based on the use of the decimal system, like carrying over ones, or multiplying with the tens first and the ones later, and the like. The child is then supposed to solve simple arithmetical problems abstractly, without imagining a collection of marbles that gets increased or diminished. The child is supposed to be able to solve problems like

(58) 26789–789+(2×23)=?

In this last case, for example, the child is supposed to see more or less directly and without much calculation that the digits in ‘789’ are the last three digits of ‘26789’ and thus subtracting ‘789’ from the latter is ‘26000’.

All this sounds easy, if not trivial to us. But in fact it is extremely hard for a child to learn this. It will take several years before children make the transition from being able to count and use natural number words to being able to complete simple arithmetical calculations. By the time children have learned to do basic arithmetic, they will already have mastered other complicated things, like talking, or making up a story, or finding their way home, or knowing what they shouldn't do. I think it should be quite surprising that learning arithmetic is so difficult. It literally takes years of hard work and repeated training for children to be able to solve simple arithmetical equations. Why is arithmetic so hard, and why does it seem so easy to us now?

4.3 Cognitive Type Coercion

The contexts in which arithmetical symbols are introduced and the examples with which arithmetical equations are illustrated suggest that arithmetical equations at first express bare determiner statements. After all, the determiners, at least for small numbers, are part of the everyday vocabulary of the students who are learning arithmetic. Such bare determiner statements can be completely understood, given what the students have mastered before entering mathematics education. To give a primacy to the bare determiner statements in the beginnings of mathematics education is not really to decide the question of the nature of arithmetic in any way. The crucial question will be how arithmetic develops from this starting point, and what it ends up as after basic mathematics education has ended. We will have to see how the singular arithmetical equations arise from a starting point of plural, bare determiner statements.

Numerical equations like

(59) 3+2=5

and basic arithmetical truths more generally are first learned in the context of thinking about the sizes of collections, and so they might in these contexts be appropriately expressed as

(60) Three and two are five.

However, thinking about arithmetic in this way, involving bare number determiners, has its cognitive obstacles, in particular when the numbers get larger. Once we try to make calculations that are not obvious any more and once we try to solve arithmetical problems of a somewhat greater complexity, we run into cognitive difficulties. Thoughts that are expressed with bare determiners and that involve operations on determiners are quite unusual. In ordinary thinking, there are only very few cases of this besides the ones involving number determiners. 'Some but not all' and a few more come to mind, but their complexity is rather limited. Number determiners are special in this respect because they allow for the expression of complicated thoughts that involve essentially only bare number determiners and operations on them. Our minds, especially when we are young children, are not very well suited to reason with such thoughts. As we have seen above, learning even fairly basic arithmetical truths and doing simple arithmetical calculations is a substantial task for a small child and takes years to accomplish. Anything that helps to solve arithmetical problems will be gladly adopted.

Our minds mainly reason about objects. Most cognitive problems we are faced with deal with particular objects, whether they are people or other material things. Reasoning about them is what our mind is good at. And this is no surprise. We are material creatures in a material world of objects, and the things that matter the most for our survival and well-being are material objects. But what precisely is the difference between reasoning with thoughts that are paradigmatically about objects, and reasoning with thoughts that are expressed with bare determiners? Why is our reasoning generally better with one rather than the other? This difference can be nicely illustrated by adopting a certain widely held picture of reasoning and the role of mental representations in reasoning. According to this view, reasoning is a process of going from one mental state to another such that what facilitates the transition between mental states does not directly operate on the contents of the mental states, but rather on representations that have these contents.[17] This process has to track certain properties of the contents of the mental states for it to be good reasoning, but since the contents themselves are not directly accessible, reasoning is a process that directly operates on the representations that have content and only indirectly on the contents, via these representations. The reasoning process thus primarily gets a grip on the representations, not by their representational features, but by their nonrepresentational features. To put this neutrally, reasoning primarily operates on the *form* of a representation, and these operations on the form of a representation have to mirror properly operations on the contents of the representations. The hypothesis that mental representations form a language of thought is one way to spell this out.[18] In this formulation, reasoning can be seen as operating on the syntax of the language of thought and not directly on its semantic features. Representations that are about objects will have a particular form, or "syntax," and representations that can be expressed with bare determiners will have a different form, all things being equal. Our reasoning ability corresponds to our ability to make the transition from certain mental representations to others, thereby preserving certain representational properties (like truth). Thus the observation made above, that our minds are better at reasoning about objects than at reasoning with thoughts that can be expressed using bare determiners, can be reformulated as follows. In reasoning our minds favor representations that have a certain form, the one paradigmatically had by representations about objects, over others that have a different form. In particular, our reasoning is more efficient when it operates on representations that have the form that is paradigmatically had by representations that are about objects.

Now, consider again the difference between the plural and the singular basic arithmetic statements:

(47) Two and two are four.

(50) Two and two is four.

We have seen above that we can understand determiners to belong to a particular type of expression, namely ((e, t), ((e, t), t)). The type of 'and' in the plural statement, correspondingly, is the rather high type mapping two determiners onto a determiner. Thus, if we abbreviate the type of determiners as 'd', then the type of 'and' in the plural statement can be represented in our type notation as (d, (d, d)), which is the type of a function that maps two determiners onto another determiner.[19] In the singular statement, the number words are of a low type, the type of objects e, and 'and' corresponds to an operation on objects that is correspondingly of the low type (e, (e, e)). These features, mutatis mutandis, will carry over to the mental representations that have the same content. In fact, we can see that the individual expressions in the singular and plural statements correspond pairwise to a "type raising" or "type lowering" of each other. The plural determiner 'two' corresponds to the singular 'two' via a type lowering, and the corresponding type lowering holds between the two uses of 'and' and the other number words. Indeed, the difference between singular and plural statements can be quite generally associated with a lowering or raising of certain types, and this feature has been used in the semantics of plural statements in natural language (see van Benthem 1991, 67ff. and van der Does 1995). That there is a systematic correspondence between the singular and the plural statements in terms of type change, together with the above story about the role of the form of a representation in reasoning, suggests the following account of the relationship between the singular and plural uses of number words in the basic arithmetic equations.

When we encounter arithmetical problems and attempt to reason about them with representations involving high types, we quickly run into cognitive difficulties. Our reasoning is not very good at working with representations having this form. On the other hand, we have very powerful resources available for reasoning, namely those that operate on representations that have the form of representations that are paradigmatically about objects. Operations on representations of this kind are well developed in creatures like us, but we can't use them to reason with thoughts that would be expressed with bare determiners and that involve higher types. We thus have a mismatch between the form of the representations that we want to reason with and the form of a representation that is required for our powerful reasoning mechanisms to be employed. But this mismatch can be overcome quite simply. We can force the representation to take on a form that fits our reasoning mechanism. The representation will have to change its form by systematically lowering the type of the bare determiners and operations on determiners to that of objects and operations on objects, respectively. Once this is done, the reasoning mechanisms we have can get a grip. This type lowering corresponds exactly to the difference between the plural and singular arithmetical statements. I will call the process of changing the type of the form of a representation to facilitate cognition *cognitive type coercion.* It is a special case of the more general phenomenon of type coercion, which can occur for a variety of reasons, not necessarily to facilitate cognition, and which can have a variety of other results, not necessarily focused on the form of a representation. I will discuss other kinds of type coercion shortly.

The process of cognitive type coercion forces a representation to take on a certain form so that a certain cognitive process can operate with this representation. Systematically lowering the type of all expressions (or the mental analogue thereof) is a way of doing this, and the difference between our ability to reason with representations involving low types rather than high types explains why this type lowering occurs in the case of arithmetic. It will occur in the process of learning arithmetic once the arithmetical problems that we are asked to solve become complicated enough that thinking about them with thoughts involving higher types becomes a cognitive burden. Note that according to the cognitive type coercion account we merely change the form of the representation. We do not replace one representation with another one that has a different content. We take the same representation and change its form so that our reasoning mechanism can operate on it. The content of what is represented remains untouched by this. To put it in terms of the language of thought, we change the syntax of a representation so that reasoning mechanisms can get a grip on these representations. Other than that we leave it the same. And what holds good for mental representations will hold good, mutatis mutandis, for their linguistic expression in language. The singular arithmetical statements are the expression in language of thoughts involving the type-lowered representations.

4.4 Contrast: Other Kinds of Type Coercion

Type coercion is the general phenomenon of something of one type being forced to take on a different type. This phenomenon is widely discussed in computer science, and it is familiar from semantics as well. In computer science, for example, an expression in a programming language can be coerced or forced to take on a certain type so that it can be interpreted in a certain situation. In semantics, sometimes an expression can be forced to be of a certain type so that a sentence as a whole becomes interpretable. Let's look at some cases of type coercion and how they differ from cognitive type coercion.

One surprisingly neglected approach to solving Frege's Other Puzzle is one that uses the idea of semantic type shifting that has been developed in natural language semantics (see Partee and Rooth 1983 and Partee 1986 for two of the classic papers in this tradition). Semantic type shifting has been proposed to solve problems in natural language semantics that arise from expressions that apparently have different types on different occasions. The multiple uses of 'and' are a good example of this. 'And' can conjoin expressions of many different types, such as sentences, verbs, or determiners, among others. Because of this, it is hard to say what type should be assigned to 'and' itself. What seems to be required is that 'and' is of a different type on different occasions. But it would be a mistake to think that these cases involve different words that are all pronounced the same way as 'and'. A better way to go is to think of 'and' as having variable type: it can take on different types on different occasions, but all the types it can take on are related in a certain way. Such a more flexible approach to assigning types to expressions will make these assignments simpler and more systematic. What type a particular occurrence of 'and' will take is left open by this proposal so far. The semantics of the word as such does not determine that the word takes a particular type, it only specifies a range of possible types and what contribution to the truth conditions the word makes in what type. Rather it is the occurrence of 'and' in a particular sentence that determines what type it takes. For example, in

(61) John sang and Mary danced,

'and' conjoins sentences, whereas in

(62) John and Mary danced together,

it conjoins noun phrases. Such an account of 'and' thus takes it to be of variable type. There is a range of types that 'and' can take, for each type that it can take, its contribution to the truth conditions is of a particular kind, and what type it takes on an occasion is determined by its occurrence in a sentence.

In general the situation is a little more complicated since there might be more expressions than just 'and' that are of variable type, and thus there might be different ways to specify these types to make the sentence as a whole come out meaningful. In these cases, there will be different readings of the sentence corresponding to different ways to specify the types.

Thus, semantic type coercion is the phenomenon where an expression of variable type is forced to take a particular type on a particular occasion so that the sentence as a whole in which it occurs is semantically interpretable. The case of conjunction, also called "generalized conjunction," is one example that illustrates this. This is a case of type coercion since the semantic type of a particular occurrence of a word of variable type is determined by or coerced by the types of the other phrases in the sentence in which it occurs. It is coerced to be the type (or one of the types) that makes the sentence as a whole meaningful.

Type-shifting principles are principles that tell us what type an expression can take and how these types relate to each other. Such principles are widely used and discussed in linguistics and semantics, but to my knowledge they have not been used to attempt to show how number words can occur both as singular terms as well as determiners. To do this, one would have to specify what types these expressions can take, how they relate to each other, and what contribution they makes to the truth conditions in that type. One could then attempt to solve Frege's Other Puzzle along these lines, using only semantic type shifting and semantic type coercion to explain the different occurrences of number words. It is somewhat involved to spell out the details of such a proposal, and to investigate whether it can give us a solution to Frege's Other Puzzle. I attempt this in Hofweber 2005c, but I won't be able to give the details here. Even though this is a very promising line of attacking Frege's Other Puzzle, it won't solve it. This is not because there is anything wrong with semantic type shifting as such. It clearly is a really good idea. But it doesn't give us the right results when it comes to number words. According to a natural extension of type-shifting principles developed in semantics, there should be readings of sentences with number words in them that clearly are not there. And it doesn't seem to be possible to modify the proposal to get the right results. Again, I won't be able to argue for this here, or even spell out the proposal in any detail. (I do this in Hofweber 2005c.)

Semantic type coercion differs from the present proposal, cognitive type coercion. Semantic type coercion claims that number words are of variable type, including the type of determiners and the type of singular terms. Because of this, number words can occur syntactically in different positions, and when they do, they have a different semantic function. Cognitive type coercion, on the other hand, does not see the occurrence of number words in singular term position as arising from the semantically variable type, but rather considers their occurring syntactically as singular terms to be contrary to their semantic type. Believers in cognitive type coercion will hold that not all syntactic occurrences of a phrase are closely associated with a corresponding semantic type. They will hold that the syntactic occurrence of a phrase in a particular position does not necessarily reflect on its semantic function. We will see a different case of this below, also in connection with number words. There we will examine a case where a determiner occurs in a syntactic position contrary to its semantic type for a different reason.

Thus, the crucial differences between semantic and cognitive type coercion are the following: semantic type coercion holds number words to be of variable semantic type, cognitive type coercion does not, at least not in the examples discussed. Semantic type coercion explains the different syntactic occurrences as a consequence of the variable type of the number words. Cognitive type coercion explains the different syntactic occurrences differently. The occurrence of number words as determiners is explained as reflecting the syntactic and semantic category of number words; the occurrence of them as singular terms is explained as being contrary to the semantic type of number words, but as occurring this way, nonetheless, for cognitive reasons.

I think that semantic type coercion is the second-best attempt to solve Frege's Other Puzzle, but once we look at the details we can see that it can't be right. I favor cognitive type coercion instead.

Another proposal that could use a similar type-lowering idea and that could motivate it in a similar way is a fictionalist or pretense proposal. According to it, we do not change the representation involving the mental analogues of bare determiners to one that has the form of a representation about objects, while leaving the content the same, as the cognitive type coercion proposal has it. Nor does this proposal hold that number words are semantically of variable type. Rather it holds that we use a different representation, with a different content, and we use pretense to connect them. One way to spell this out is to say that we pretend that there are numbers and operations on them that exactly correspond to the operations on bare determiners. The number words that refer to these objects, on such a proposal, will however not be determiners, which after all don't refer, but rather new words, names of the pretended numbers that are pronounced the same way as the number determiners. This proposal is distinctly different from the two proposals discussed above. The above two do not involve pretense. In the case of generalized conjunction, we do not pretend that 'and' conjoins verbs as well as sentences, it does conjoin both kinds. And in cognitive type coercion, we do not pretend that the representations have a different form, they do have a different form. This is not the place to criticize such a fictionalist proposal, of course, but rather to contrast it with both the present view and semantic type coercion.[20]

Simply put, we can say that the three proposals make the following three different suggestions about how number words relate to each other in the singular and plural arithmetic statements: a) semantic type coercion holds that the number words in the singular and plural arithmetic statements are the same word, but that they are semantically and thus syntactically of variable type, b) cognitive type coercion holds that the number words in the singular and plural arithmetic statements are the same word, with a fixed semantic type, but with a different syntactic occurrence that reflects a cognitive need, and finally c) the fictionalist proposal holds that the number words in the singular and plural arithmetic statements are different words and the relationship between the singular and plural statements involves pretense.

Relating number symbols to numerical quantifiers via type shifting is not an unfamiliar idea in the philosophy of mathematics. Harold Hodes has made this connection in a series of papers, see, for example, Hodes 1984 and Hodes 1990. Although the position defended in the present article is congenial to Hodes' overall view, there are a number of differences. For Hodes, the relationship between a number and the corresponding number quantifier is that of encoding or representing. Hodes, in particular, in Hodes 1984, likens numbers to fictional objects that are posited as representers or encoders of number quantifiers, and his position there was labeled "coding fictionalism." This fictionalist aspect of Hodes' view is less congenial to the present proposal than other aspects of his view. Particularly congenial is his view that the syntactic formulation of arithmetical statements involving number words as singular terms does not bring with it a semantic interpretation that has an independent domain of numbers as particular objects. In Hodes 1990, he considers different formal languages that capture number words as singular terms and number quantifiers, develops a model- theoretic semantics for them, and investigates their model-theoretic relationships. Hodes' proposal and the present one agree that the syntactic form of arithmetical statements does not bring with it a semantic interpretation that takes these statements to be about a domain of objects. And both proposals agree in their association of number words as singular terms with number quantifiers, or determiners in our case, through type shifting. They disagree in how such a connection is established. Hodes primarily focuses on formal languages and their model-theoretic semantics, whereas in the present proposal, I primarily focus on natural language and cognition. Further disagreements concern the function of number words as singular terms; Hodes takes these number words to function as encoders of number quantifiers, but according to the present proposal, they are still number determiners. The association of number quantifiers or determiners with number words as singular terms through type shifting is central to both proposals.

The cognitive type coercion proposal gives us an account of the relationship between the singular and plural arithmetical equations, and it explains, at least in outline, why thinking about numbers is a lot like thinking about objects. According to this account, it is only like thinking about objects when it comes to the form of the representation involved in thoughts about numbers, but these thoughts are not about any objects since their content is the one expressed with the bare determiner statements. This, so far, is not an account of all uses of number words, but only one step toward such an account. For example, it doesn't directly carry over to the singular-term use that we discussed at the beginning of this article, as in (3). We will look at this below, but first, let's sum up.

4.5 Summary of the Proposal So Far

The proposal made so far aims to explain the connection between the plural and singular readings of arithmetical statements, and how they relate to the arithmetical symbolism. It gives a unified account of many, though not all, uses of natural number words. According to this proposal, plural arithmetical statements are formulated with (semantically) bare determiners. These determiners are an ordinary part of natural language. The use of these determiners plays a central role in the introduction of the mathematical symbolism when basic arithmetic is introduced and learned. However, there are cognitive obstacles to arithmetical reasoning that relate to the unusually high type of these determiners. To overcome these difficulties, a child that is faced with more and more complex arithmetical problems will adopt a systematic type lowering at the level of mental representation to employ reasoning mechanisms that require representations of this lower type. This type lowering we called *cognitive type coercion* because representations of a certain type are forced into a different type for cognitive reasons. This type coercion is at the level of the form of the representation that represents basic arithmetical truth, not at the level of the content of what is represented. The content remains untouched and is the same as the content of the plural basic arithmetical statements that are formulated with bare determiners. Thus, both the singular and the plural reading of symbolic equations are correct. The plural one is the natural expression of the bare determiner statement; the singular one is the linguistic expression of the cognitively type-coerced representation. This account solves a special case of Frege's Other Puzzle. It explains how number words, in this limited set of examples, can syntactically occur both as determiners and as singular terms. According to this account, one and the same number word is used on both occasions; number words are determiners, but for the reasons spelled out above, they can also occur syntactically as singular terms.

The proposal so far has not been the result of a purely a priori reflection on the nature of mathematics, or on what theorems can be proven in what formal system. Essential parts of what I have proposed could be characterized as being largely empirical claims. That cognitive type lowering gives us an advantage in reasoning, that children learning arithmetic in the way I've outlined, and a number of other things I have suggested are claims about humans and what they do. We can't decide a priori if these claims are correct, and thus the fate of the proposal in this article depends in part on how things turn out empirically. I know that, in the philosophy of mathematics in particular, some philosophers will see this as a flaw of the present approach, especially since the empirical questions on which this view ultimately depends are not settled in this article. The philosophy of mathematics is still a discipline where a theorem is considered the clearest sign of progress and speculative empirical considerations are best left for someone else. This is unfortunate since many views discussed in the philosophy of mathematics depend on the answers to empirical questions and ultimately on what people do. Take fictionalism for example. Whether or not fictionalism is a route to nominalism might be a purely philosophical and a priori question. But whether or not fictionalism about arithmetic is true is an empirical question. For it to be true, we who do arithmetic have to engage in a pretense or some other cognitive stance, depending on the view. No purely a priori philosophical speculation can determine whether we engage in such a pretense when doing mathematics. Answering Frege's Other Puzzle similarly depends on what we do when we use number words. The present proposal offers a solution to the puzzle based on a view about what we do with number words and why we do it. This view has empirical support, although it might turn out to be false on empirical grounds. I think it is our best bet, though. So far at least.

5. The Number of Moons of Jupiter Is Four

We started out by noting that there are three different uses of the expression that is pronounced 'four'. The first was the adjectival or determiner use, as in 'four moons'. The second was the singular-term use, as in

(3) The number of moons of Jupiter is four,

and the third is the symbolic use, as in '4'. Above, we directly connected the symbolic use with the determiner use. We claimed that the meaning of the symbolism can be traced to the determiner use of number words. A Frege-style analysis of this situation sees it differently. For a Fregean, '4' is a singular term that stands for an object. That numbers are objects can be seen, they argue, from the occurrence of singular terms that stand for numbers in true identity statements like (3). Symbolic numerals can be seen simply as singular terms standing for the same objects as do number words. So, the Frege-style analysis takes symbolic numerals to be derivative on singular-term uses of number words. The present account, to the contrary, takes symbolic numerals to be derivative on determiner uses of number words.

I mentioned above that there often isn't much attention paid in the philosophy of mathematics to determiner uses of number words since the resulting quantifiers are first-order expressible. And the Frege-style analysis of the situation usually doesn't give one much of a story about number words used as determiners. A complete account of number words and how they relate to mathematics will have to include an account of all three of these cases and how they relate to each other. The Frege-style analysis, at least the versions I know of, falls short in this respect.

A similar charge can also be made against the present account. We have not said much about the singular-term use of number words, at least outside of singular basic arithmetical statements. Unless these other uses of number words as singular terms can be dealt with as well, our solution to Frege's Other Puzzle is at best partial. Thus, so far we have no complete account of how one and the same word can occur in such different positions, sometimes as a determiner, sometimes as a singular term. The account so far does not cover the singular-term occurrence of 'four' in (3). A closer look at these examples, however, reveals that they are in fact compatible with our solution to Frege's Other Puzzle, but for a quite different reason. The pair (1) and (3) is puzzling in at least two ways. First, the word 'four' appears in two different syntactic positions, which is a special case of Frege's Other Puzzle. Second, besides this, the two sentences seem to be obviously truth conditionally equivalent. But how can they be obviously equivalent, given that the number words in them are apparently so different?

In this section, I would like to outline an account of the relationship between (1) and (3). This issue deserves a more thorough investigation than can be given in this article. I spell out all the details of what I outline below in Hofweber 2005b. In addition, in what I will outline, I will cover only the case of the relationship between sentences like (1) and

(3). It will not be a general account of the singular-term use of numerals. It will still leave open what is going on in certain other uses of number words as singular terms in statements that are neither singular basic arithmetical equations nor of the same kind as (3). A trivial example is

(63) Two is a prime number.

I will not be able to discuss these uses here, but leave them for another occasion.

The key to understanding the relationship between (1) and (3) is to see whether or not they have different uses in communication. If they are truth conditionally equivalent, as I grant they are, is there any other relevant difference between them? Do they have a different effect on a discourse? If so, maybe we can understand their relationship better through analyzing the different effects they have on a discourse.

When we look at the role of (3) in communication, we can see that the relationship between (1) and (3) is in certain ways analogous to the relationship between a regular subject-predicate sentence, like

(64) Johan likes soccer,

and what is called a *clefted* sentence, like

(65) It is soccer that Johan likes,

or

(66) It is Johan who likes soccer.

All these sentences have the same truth conditions and communicate the same information, but there is a clear difference between them. (64) communicates that information neutrally, with no particular aspect being stressed or emphasized. (65), however, puts an emphasis on what is communicated. (65) stresses that it is soccer that Johan likes and contrasts it with other things. What this contrast class is will depend on the context of the utterance. Similarly, (66) contrasts Johan with other contextually salient people and claims that Johan is the one who likes soccer. The common term for this phenomenon is *focus*. Focus is usually achieved through intonation. By phonetically stressing a word or phrase, one can get similar focus results. However, in the cleft construction, which is used in (65) and (66), the focus effect does not arise from intonation but from the sentence structure. It is not *intonational focus* but rather *structural focus*.[21]

These examples illustrate a general phenomenon. Sometimes two sentences have the same truth conditions, and obviously so, but differences in the underlying syntactic structure have the result that one of the sentences in this pair expresses information neutrally, while the other one does so with a focus on one or another aspect of what is communicated. And this general phenomenon, I claim, also applies in our case of the relationship between (1) and (3). The latter brings with it a focus effect independent of intonation, whereas the former does not. To see this, just consider the difference such utterances have in communication. For example, if you ask me what I had for lunch and I answer

(67) The number of bagels I had is two,

this reply would be very odd. However, had I answered

(68) I had two bagels,

this would have been perfectly fine. The difference between the two is that in the first I focused on how many bagels I had not on what I had. You, however, asked me about what I had not about how many I had. Had you asked me how many bagels I had, then (67) would have been acceptable.

In Hofweber 2005b, I argue that this focus effect can't be explained if one thinks that (3) is both syntactically and semantically an identity statement with two (semantically) singular terms. But it can be explained if (3) has a different syntactic structure, one that results from extracting the determiner and placing it in an unusual position that has a focus effect as a result. Thus, in (3) 'four' is a determiner that has been "moved" out of its usual position. This is a particular case of how syntactic structure can give rise to focus effects. Even though I won't be able to give the details of this account here, the upshot is of importance for our discussion. It implies that even in (3), 'four' is a determiner and not a referring expression. In particular, the word 'four' is the same in both (3) and (1). The reason for the occurrence of 'four' as a singular term in (3) and the reason for its occurrence as a singular term in a singular arithmetical equation are thus different and independent. The occurrence of '4' or 'four' as a singular term in arithmetical equations is the result of cognitive type coercion. The occurrence of 'four' as a singular term in (3) is the result of extraction of the determiner for a structural focus effect.

Given this account of 'four' as it occurs in (3), we can see that all three uses of number words, the singular-term, determiner, and symbolic uses, fit together into one uniform story. This story gives primacy to the use of number words as determiners and shows how the other uses are based on it. In neither case are the number words referring expressions or "semantically singular terms," that is, expressions that have as their semantic function to pick out some object or entity.

6. The Philosophy of Arithmetic

Suppose that what I have said so far is more or less correct. What would follow for the philosophy of arithmetic? To answer this, we first have to note that so far we have covered only basic arithmetical statements, ones that contain no quantifiers and no variables. So we should distinguish two kinds of questions: first, what follows from the present account for the philosophy of arithmetic for basic arithmetical statements; second, can this account be extended to quantified, full arithmetic?

6.1 Basic Arithmetic

If basic arithmetical statements involve only bare determiners and operations on them or comparisons between them, then such statements involve no referring terms and, as we discussed above, are true no matter what objects exist. In fact, using a notion of logicality that covers higher-order expressions, like van Benthem 1989 or McGee 1996, we can argue that basic arithmetical statements are logical truths. In any case, basic arithmetical statements have many of the features that made logicism about arithmetic attractive: they are true no matter what objects exist and their truth is necessary and objective. The connection of the present view to logicism is further explored in Hofweber 2005e. In addition, the account of basic arithmetical statements explains why arithmetic seems to be about objects, even though it really isn't. The form of the representations (mentally as well as linguistically) is that of representations about objects, although the content isn't about any objects. Thus the present proposal about basic arithmetic suggests a view of it according to which such arithmetical statements are objectively true no matter what objects exist. Before we move on, we should consider a well-known objection against an apparently similar view and see why it does not carry over to the present proposal.[22]

6.2 Contrast: The Adjectival Strategy and Second-Order Logic

The proposal about basic arithmetical statements defended here is reminiscent of what Dummett (1991, 99ff.) has labeled "the adjectival strategy." The adjectival strategy claims that number words in arithmetic are derivative on their adjectival use in natural language, and it is contrasted with "the substantival strategy," which claims that their use in arithmetic is derivative on their singular-term use. Frege was a defender of the substantival strategy, whereas my view is in the ballpark of the adjectival strategy. The adjectival strategy, as it is commonly carried out, however, has a well-known problem, and it might seem that this problem carries over to the view defended here. In this section, I will contrast this common way to spell out a version of the adjectival strategy with the position defended in this article and argue that the objection to the former does not carry over to the latter.

In the philosophy of mathematics literature, the preferred way to spell out the adjectival strategy is to map sentences written with arithmetical symbols, like

(69) 5+7=12

onto sentences in second-order logic, like

(70) $\forall F \forall G(\exists_5 x F(x) \wedge \exists_7 y G(y) \wedge \neg \exists z(F(z) \wedge G(z)) \rightarrow \exists_{12} w(F(w) \vee G(w)))$

whereby the number quantifiers, like '$\exists_5 x$', are abbreviations of blocks of first-order quantifiers, in the usual way. Let's call this the *second-order logic strategy*, or *SOL strategy*, for short. How precisely the relationship between (69) and (70) is supposed to be understood deserves further discussion. Does (70) make the underlying logical form of (69) explicit? Is one just a replacement for the other with the same truth conditions? No matter what one says here, any claim that (70) spells out the truth conditions of (69) faces the following serious objection, which we will call *the objection from finite domains*. The second-order sentences that are supposed to correspond to the arithmetical ones will not even get the truth values right if there are only finitely many objects. If there are only *n*-many objects and the arithmetical statements involve numbers larger than *n*, then the antecedent of the second-order statement is false and the whole statement is thus vacuously true. Therefore, if there are only finitely many objects, arithmetical equations with numbers larger than the number of objects will all be vacuously true, no matter what they say about addition, an absurd consequence. One way that defenders of the SOL strategy have attempted to get around this is to involve a modal version of statements like (70).[23] But this modal version has its problems. In particular, it bases arithmetical truths on modal truths, which intuitively seem much more elusive.

It is important to note how the SOL strategy is different from the present proposal. This difference will make clear that the objection from finite domains, which is a very good objection against the SOL strategy, does not carry over to my proposal.

The objection against the SOL strategy is a good objection, as long as the SOL strategy holds that the truth conditions of the arithmetical equations are captured by the second-order statements. Under certain conditions, when there are only finitely many objects, the truth values of some arithmetical statements and their corresponding second-order statements come apart, and thus their truth conditions can't be the same. '5+7=10' is always false, but the corresponding second-order statement is true under the condition that there are only four objects. Thus the objection from finite domains is a good objection against any view that holds that the arithmetical statements are truth conditionally equivalent to the second-order statements. It would carry over to my proposal if it were true that the truth conditions of the bare determiner statements, in this case

(71) Five and seven are twelve,

are the same as the truth conditions of the second-order logic statements, in this case (70). Thus, to extend the objection against the SOL strategy to my proposal, one would have to argue that the bare determiner statements are truth conditionally equivalent to the second-order logic statements. But I think we can see quite clearly that this is not so. Not only is there no good reason that these should be equivalent, but it seems clear upon reflection on some examples to be discussed shortly that they are not equivalent, and that no objection similar to the objection from finite domains will carry over to my proposal.

One way in which one might argue that (70) has the same truth conditions as (71) is to argue that the former spells out the underlying logical form of the latter. I take 'logical form' here in a sense in which the logical form of a sentence is semantically revealing. That is to say that the logical form of a sentence makes certain semantic features of the sentence explicit, ones that might have been left implicit in the natural language expression of this sentence. Given this conception of logical form, the second-order statements do not articulate the logical form of the bare determiner statements. Many important semantic features of the latter are not captured by the second-order logic statements, and the second-order logic statements bring in several things that are not in the bare determiner statements. For example, the 'and' in (71) is a collective operation on plural bare determiners. In standard second-order logic, there is no room for plurals, there are no collective operations or bare determiners, and conjunction is restricted to formulas, that is, to sentences and predicates, but it isn't defined for terms. All this semantic structure is misrepresented in (70). (70) is of conditional form, but (71) is not. And (70) treats number quantifiers as syncategorematic. We have seen above that this, too, is to be rejected as a proposal about the logical form of number determiners. So (70) does not make the underlying logical form of (71) explicit. (70) is as good as it gets *in second-order logic*, but that just means that second-order logic is not up to the task of spelling out the logical form of (71).

That (70) does not capture the logical form of (71) does not establish that they do not have the same truth conditions. But we can see quite independently that the truth conditions of these statements are different. As we have seen, the second-order logic statements depend on the size of the domain for their truth value, in particular, if this size is finite, then many of them become vacuously true, even though they correspond to false arithmetical equations. To see that the truth value of the bare determiner statements does not likewise depend on how many things there are, we should look at some examples. Here I am simply asking you to judge the truth value of ordinary English sentences. Consider:

(72) Two dogs are more than one,

which is clearly true. Does its truth depend on the existence of dogs? To see that it doesn't consider

(73) Two unicorns are more than one.

It is also true, even though there are no unicorns. And

(74) Two unicorns are more than three,

is false, even though there are no unicorns. More generally, the ordinary English sentence

(75) Two are more than one,

is true, but its truth is not dependent on how many things there are, and

(76) Two are more than three,

is false, and its falsity also does not depend on how many things there are. Similarly, the English sentence

(77) Five unicorns and seven more are twelve unicorns,

is true and

(78) Five unicorns and seven more are ten unicorns,

is false, even though there are no unicorns. And more generally,

(71) Five and seven are twelve,

is true no matter how many objects there are, and

(79) Five and seven are ten,

is false no matter how many objects there are. I base these claims on judgments about the truth value of the ordinary English statements. As such I find them hard to dispute. Thus, the truth conditions of the bare determiner statements are not captured in the second-order statements, since the truth values of the latter depend on how many things there are, but that of the former do not.

All this is not to say that we cannot capture the truth conditions and logical form of the bare determiner statements in a formal language. Some of the things that are lacking in standard second-order logic, namely plurals, collective operations on determiners, and so forth, can be found in other formal languages, for example, in a type-theoretic framework with a proper modeling of plurals. But any argument about which one of these models captures the truth conditions of (71) will have to be carried out at the level of natural language. We first have to understand what the truth conditions of (71) are, and then we have to see which formal language is expressive enough to capture them and how to capture them in that formal language. Standard second-order logic alone won't do. The objection from finite domains, which is a serious problem for the SOL strategy, thus does not carry over to my proposal. To the contrary, the truth and falsity of the bare determiner statements do not depend on what there is nor on how many things there are.

6.3 Full Quantified Arithmetic

Now there is one large issue left for us to address: can the account of quantifier-free arithmetic given above be extended to quantified arithmetic, and if so, which of the properties of quantifier-free arithmetic carry over to full quantified arithmetic. I wish I had the space here to address this issue properly, but I will have only the opportunity to outline a view that, if correct, will allow us to extend our solution to Frege's Other Puzzle into a philosophy of arithmetic. I will have to refer again to other work for more details and arguments. But it should be of interest nonetheless to see how what we have seen so far fits into a larger picture.[25]

It is not easy to see how quantifiers interact with number words, assuming what we have said about them above is correct. It will not be sufficient to look at quantifiers in formal languages, like first-order quantifiers. To see how quantifiers in formal languages interact with arithmetical statements, we will first have to represent these arithmetical statements in the formal language. And for that it will matter what properties of the arithmetical statements we want to capture in the formal language. For example, I see no problem in representing basic arithmetical statements in a first-order language that takes numerals to be terms, and to give such a language a model-theoretic semantics, where the terms denote an object in the domain, even though natural number words are not denoting expressions. When we represent arithmetic formally we do not primarily care about representing these semantic features of number words. Rather, we are interested in capturing the inferential relations of arithmetical statements to one another. This can be done elegantly in first-order logic, even though the referential feature of number words is not correctly modeled this way. But neither is the noun phrase-verb phrase structure of ordinary English sentences, including arithmetical ones, but this, too, is not what we aim to model. To see how we should understand the interaction between number words and quantifiers, we thus have to look at natural language quantifiers.

For a number of reasons, quantification in natural language is quite a bit more complicated than in formal languages. One, but only one, of the extra complications arises from the interaction between quantifiers and syntactically singular terms that do not denote or refer. These terms don't have to be number words, as we understood them above. There are a variety of other candidates, although most are controversial in at least some philosophical circles. In some recent articles (Hofweber 2000a, 2005a, 2006), I have defended the following view of this aspect of quantification in natural language. Quantifiers have at least two functions in communication. One is to range over a domain of entities, whatever entities exist, or a contextual restriction thereof. This reading of quantifiers is well captured in the usual model-theoretic semantics of quantifiers. Another function quantifiers have, however, is to occupy a certain inferential role. Such an inferential role relates statements with quantifiers in them to other statements that are the instances of the quantified statement. Paradigmatically, the particular quantifier occupies the inferential role that makes the inference from 'F(t)' to 'Something is F' valid for whatever 't' may be. We have a need for both of these readings of quantifiers in everyday communication, and in languages like ours, in particular languages that contain syntactically singular terms that do not aim to stand for any entity, these two readings of the quantifiers come apart in their truth conditions. No one contribution to the truth conditions can give you both, the domain conditions and the inferential role. This does not mean that quantifiers are lexically ambiguous and that there are two quantifiers pronounced 'something' in natural language. The proper way to understand this is to consider it a case of semantic underspecification. Semantic underspecification is a widely occurring phenomenon in natural language. A particular expression might not fully determine what contribution to the truth conditions it will make when it occurs in an utterance. The contribution that the semantics of that expression makes to the truth conditions of the utterance does not then fully specify the truth conditions of the utterance. Quantifiers, according to this view, are semantically underspecified, and on different occasions of utterance, they can make a contribution to the truth conditions that gives them either a certain inferential role or certain domain conditions. These two contributions to the truth conditions that quantifiers can make are not unrelated. In fact, they coincide in the limit, when we deal with simple languages and simple worlds. In a situation where every term denotes an object and every object is denoted by some term, inferential role and domain conditions coincide with respect to truth conditions. But they do come apart in natural languages like ours where some terms do not denote anything and some things are not denoted by any term.

Suppose this view of natural language quantifiers is correct.[26] It has a number of interesting consequences. For one, it implies that basic quantified statements like

(80) There is a number between 6 and 8.

have at least two readings, one where the quantifier is used in its inferential-role reading, the other where it is used to impose a domain condition. And similarly, there will be two readings of

(81) There are numbers,

and

(82) Are there numbers?[27]

These readings will in some ways resemble Carnap's distinction between internal and external questions about what there is, even though both questions are fully meaningful.[28] Internal questions will have trivial affirmative answers, whereas external questions are the ones that are the questions of ontology. However, and contrary to Carnap, external questions are just as legitimate and meaningful as internal questions, although they are generally harder to answer. This view about quantification allows for the formulation of two large-scale views about the function of talk about certain things, like numbers, or properties. These two kinds of views can be formulated for any kind of discourse, but we will focus here on talk about natural numbers. One view is *internalism* about (talk about) natural numbers. Internalism about numbers holds that quantifier-free number statements are formulated with nonreferring terms and that quantification over numbers is quantification in its inferential-role reading that merely generalizes over the quantifier-free instances. The other view is *externalism* about (talk about) natural numbers, which takes such talk to be about a domain of entities that number terms refer to and number quantifiers range over. Which one of these two is correct will largely depend on what speakers do when they talk about numbers. If the view about quantification outlined here is correct, then both views are coherent positions. To decide between them involves looking at the details of what we do when we talk about numbers. In particular, it involves a close investigation into the function of quantifier-free talk about numbers.

The view about quantification outlined above and defended in the papers referred to together with the account of quantifier-free arithmetic developed in this article form the basis for an internalist conception of arithmetic. According to it, arithmetic is not about a domain of entities, not even quantified arithmetic. Quantifiers over natural numbers occur in their inferential-role reading in which they merely generalize over the instances. The truth conditions of the internal uses of the quantifiers, which I could not motivate or outline in this article, but which I spell out in the papers cited, are such that the large-scale philosophical points about quantifier-free arithmetic carry over to quantified arithmetic on the internalist conception of quantified arithmetic. The crucial step in the defense of this view of arithmetic is the account of quantifier-free arithmetic defended in this article. And it is motivated by the solution to Frege's Other Puzzle that I have defended here. If this solution, and not Frege's, is correct, then we have the prospect of defending a position in the philosophy of arithmetic that is partly like Frege's in that it is a form of logicism that affirms the objectivity and literal truth of arithmetic, but that is also partly unlike Frege's in that it does not conceive of arithmetic as being about a domain of objects.

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Notes

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[1] Singular terms are not a precise grammatical category, and it is thus not clear if all singular terms have the same semantic function. Some philosophers have argued that they do, and there is some controversy surrounding the notion of a singular term in the philosophical literature, in particular about whether the notion is primarily a syntactic or a semantic one. See Hale 2001a and 2001b for more on this issue. We do not need to take sides in this debate here. I take the category of a singular term to be one that has paradigmatic instances in proper names and that might or might not be able to be precisely characterized. The category of an adjective also has some controversial cases, as we will see below.

[2] Here, symbolic uses of number words are ordinary mathematical uses of them. These have to be distinguished from what could be called *formal* uses. The latter are expressions in an artificial language, expressions that are, for example, also pronounced 'four', such as terms in first-order Peano Arithmetic. It is a further question how formal uses relate to symbolic uses. I address this issue below as well as in Hofweber 2005d.

[3] The original is:

> Da es uns hier darauf ankommt, den Zahlbegriff so zu fassen, wie er für die Wissenschaft brauchbar ist, so darf es uns nicht stören, dass im Sprachgebrauche des Lebens die Zahl auch attributive erscheint. Dies lässt sich immer vermeiden. Z.B. kann man den Satz 'Jupiter hat vier Monde' umsetzen in 'die Zahl der Jupitermonde ist vier'. (Frege 1884, §57)

[4] This gives the reading of 'two men' as 'at least two men'. The 'exactly two men' reading is, of course, also first-order expressible.

[5] The view that number words are syncategorematic when they are part of quantifiers, and that these quantifiers are unproblematic since they come down to first-order quantifiers is rarely explicitly discussed at any length, though widely assumed. See, for example, Hodes 1984, or Field 1989, and many others.

[6] To call these first-order quantifiers is not to contrast them with higher-order quantifiers, but to label them as the quantifiers that are part of the first-order predicate calculus, which is commonly seen as being part of logic. We will use 'first-order quantifier' in this sense in the following, unless stated otherwise.

[7] Of course, such incomplete descriptions like 'the man' bring up a variety of other issues, for example, how they are contextually completed and the like. We can ignore this for our discussion here.

[8] I use the term 'logical form' here as a level of representation that reveals certain semantic facts about sentences, and these semantic facts can go beyond facts about the truth conditions. They can include issues of scope, what the relevant semantic parts are, and more. Thus two sentences with the same truth conditions can have different logical forms. In addition, the logical form of a sentence should be free of syncategorematic expressions, if there are any. Other notions of "logical form" exist in the literature, but we do not have to investigate this issue further as long as the above remarks are kept in mind.

[9] See Keenan and Westerstahl 1997 for a survey of work that has been done in this area.

[10] This is merely for convenience and does not involve a metaphysical view of properties. We can just directly say "the semantic value of X is a function of type (e, t)" instead of "the semantic value of X is a property." In a more realistic account, we would have to consider intensional types.

[11] The use of 'refer' in the above comment contrasts with that of some philosophers who simply use it for the relationship that holds between an expression and its semantic value. On such a use, almost all words and phrases are referring expressions, and even parts of words can be referring expressions. But on our use, we mark the intuitive difference between some words that have the function of contributing objects to the content of an utterance, paradigmatically names, and others that don't have this function, paradigmatically

words like 'by' or 'very'. 'Reference', as a technical philosophical term, is used in different ways, and the present use is one common use of this word, but not the only one.

[12] See Gamut 1991 or Barwise and Cooper 1981 for more on generalized quantifiers. Further issues arise about plurals. See, for example, van der Does 1995 for both a survey and an account of various issues about plural quantifiers. I will ignore many of the real complications about plurals in this article, but van der Does (1995) provides an account that in certain ways is congenial to the view taken below. I in particular ignore the issue of different types for plural and singular quantifiers for reasons of simplicity in presentation.

There is some controversy about whether number words in the relevant uses are determiners or adjectives. This is mainly an issue of classification and not of central importance for us here. All the arguments provided below will be motivated by examples, and these arguments are valid whether number words are determiners or adjectives. I take them to be determiners. Proponents of the view that number words are adjectives usually point to some disanalogies between number words and other determiners. For example, 'three men' can combine with 'the', which many other determiners can't:

(17) The three men who entered the bar got drunk.
(18) *The some men who entered the bar got drunk.

However, some other determiners exhibit the same behavior, for example

(19) The many children who died in the war will not be forgotten.

In addition, not all determiners behave syntactically the same way. See Barwise and Cooper 1981 for cases of this and a well-known attempt to explain the different behavior semantically. Number words also behave differently than classic adjectives. For example, they fail the "seem-test" that can be used to demarcate adjectives (thanks to Randall Hendrick for pointing this out to me):

(20) They seem green.
(21) *They seem four.

However, the classification issue is not of central importance for the overall debate here. What ultimately matters for our discussion is that number words in their determiner use can form complexes, as will be discussed shortly, and that they are not themselves referring expressions in this use. Whether they are, in the end, adjectives, determiners, or form a separate class of their own is secondary.

[13] There are, of course, a number of further possibilities that also seem to be right, like

(53) Two and two make/s four.

[14] For a survey of a number of empirical issues related to the acquisition of mathematical competence, see, for example, Dehaene 1997.

[15] Others are to understand the use of numbers in ordinary life, for example, to understand the meaning of a '20' on a bus, a cake, a house wall, a dollar bill, and so on.

[16] See, for example, Verschaffel and Corte 1996, 112f. for a classification of such exercises, and also Becker and Selter 1996.

[17] This view, often called the representational theory of mind, is widely discussed by various authors, accepted by many, but is, of course, controversial. See, for example, Fodor 1990 and 1987 for discussions with further references.

[18] See the references in the last note for more.

[19] Or more explicitly, it maps a determiner onto a function from a determiner to a determiner. Taken together, it is a function that gives you a determiner when you put in two determiners, thus a function from two determiners to a determiner. Since we don't directly have two-place functions, we take one argument at a time.

[20] Unfortunately, contemporary fictionalists usually do not consider Frege's Other Puzzle as a puzzle about natural language, and they tend to understand number determiners as syncategorematic. See Field 1989 and Yablo forthcoming. But a fictionalist proposal can be formulated naturally to fit easily into our framework here.

[21] The distinction is about the source of the focus effect, not about kinds of focus. See Rochemont and Culicover 1990 for more on structural focus.

[22] In our discussion above, we focused on addition and did not explicitly discuss multiplication, subtraction, and basic comparison with, for example, 'less than'. For all of them, the issues are parallel to the case of addition, with some subtle differences. However, our main conclusions carry over to these basic arithmetical statements.

[23] See Hodes 1984 for such a proposal. For his later view, see Hodes 1990, which instead requires infinite models for the relevant arithmetical languages. This view thus holds that arithmetic requires the existence of infinitely many objects. See also Hellman 1989.

[24] See, for example, van der Does 1995 for some of the details. To spell this out properly, we will have to consider some of the issues that we mainly ignored here, like the proper representation of collective readings of plurals, the semantic type raising that van der Does holds this comes with, and other issues.

[25] The following larger picture focuses on the philosophy of arithmetic. I discuss one way in which it fits into a philosophy of mathematics that goes beyond arithmetic in Hofweber 2000b.

[26] Again, I have argued for this in Hofweber 2000a, 2005a, 2006 but won't be able to repeat the arguments here.

[27] Assuming, of course, that these sentences in fact contain quantifiers, as is widely assumed, but this deserves further discussion. In these cases, it is plausible that there is a plural quantifier involved, as in the sentence 'There are numbers between 100 and 200.'

[28] Carnap's discussion can be found in his famous 1956 essay. I discuss how the present account relates to Carnap's in Hofweber 2005a.